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# Synchronization of coupled systems with spatiotemporal chaos

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We argue that the synchronization transition of stochastically coupled cellular automata, discovered recently by Morelli *et al.* [Phys. Rev. E **58**, R8 (1998)], is generically in the directed percolation universality class. In particular this holds numerically for the specific example studied by these authors, in contrast to their claim. For real-valued systems with spatiotemporal chaos such as coupled map lattices, we claim that the synchronization transition is generically in the universality class of the Kardar-Parisi-Zhang equation with a nonlinear growth limiting term. [S1063-651X(99)50303-6]

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Since the pioneering work of Fujisaka *et al.* [1–4] and others [5–7], synchronization of chaotic systems has become a very intensely studied subject, partially due to hopes that this could lead to applications in control and secure communications [8]. In spatially extended systems, synchronization can appear in (at least) two forms. On the one hand, one can ask whether distant regions in a single such systems can oscillate in phase. After this phenomenon was observed by Chaté and Manneville [9] in high dimensional cellular automata, it was realized that it can be mapped onto the Kardar-Parisi-Zhang (KPZ) problem of the growth of a random surface [10], with a synchronized state corresponding to a globally smooth phase $\phi=\phi(\mathbf{x})$ [11–13].

In the present paper we shall deal with another problem, namely that of mutual synchronization of two identical locally coupled systems. If these systems are described by state variables $x_i^t$ and $y_i^t$ [for simplicity we assume here discrete one-dimensional (1D) space $i$ and discrete time $t$], we write the evolution in general as

$$x_i^{t+1}=f(\dots x_{i-1}^t,x_i^t,x_{i+1}^t\dots)+\epsilon g(x_i^t-y_i^t),$$
$$y_i^{t+1}=f(\dots y_{i-1}^t,y_i^t,y_{i+1}^t\dots)+\epsilon g(y_i^t-x_i^t). \quad (1)$$

The function $f$ is nonlinear such that the evolution is chaotic for $\epsilon=0$. Due to sensitive dependence on initial conditions, $\mathbf{x}^t$ and $\mathbf{y}^t$ will be completely uncorrelated in this case, unless they started with identical initial conditions. Synchronization should only be expected for $\epsilon>0$ if $g(x)$ is negative for positive small $x$, so that any small difference $x_i^t-y_i^t$ will be damped by the last terms in Eq. (1).

For chaotic systems with a finite number of degrees of freedom, there is a finite synchronization threshold $\epsilon_c$, with intermittent behavior and ''riddled'' [14] attractor basins near $\epsilon=\epsilon_c$ [1–4,16]. Recently, it was found numerically that essentially the same phenomena occur in spatially extended systems [17–19]. While chaos in systems with few degrees of freedom requires $x^t$ to be real valued, spatiotemporal chaos can occur also in systems with discrete $x_i^t$, so-called cellular automata (CA). Therefore, one can ask whether the phenomenon of mutual synchronization can occur also in the latter [15], and whether there are universal scaling laws at the synchronization threshold that apply both to real-valued systems (coupled map lattices and partial differential equations) and to CA. While the former question was asserted positively in [17], we shall argue that the latter has a negative answer. For CA, the synchronization threshold is generically in the directed percolation universality class [20], while it is for continuous systems in the universality class of KPZ growth with a nonlinear growth limiting term [20–23].

Let us first study the case of 1D cellular automata. The specific system studied in [17] was two copies evolving according to Wolfram's [24] rule 18 with periodic boundary conditions, and endowed with an additional stochastic coupling term. In rule 18, $x_i^t$ can assume two values 0 or 1, and the evolution function $f$ depends only on $x_i^t$ itself and its two nearest neighbors, $f(0,0,1)=f(1,0,0)=1$ and $f(x_{i-1},x_i,x_{i+1})=0$ else. The coupling was realized as follows: after applying the above rule to both $\mathbf{x}$ and $\mathbf{y}$, it was checked whether $x_i=y_i$. If not, a random number is drawn uniformly from [0,1]. If this number is less than some fixed number $p$, a second random number is drawn and, depending on that, either $x_i$ is put equal to $y_i$ or $y_i$ is put equal to $x_i$. Thus $x_i=y_i$ is enforced with probability $p$, while both $x_i$ and $y_i$ are left untouched with probability $1-p$. It was found numerically that the system synchronizes for $p>p_c=0.193\pm0.001$. For $p<p_c$ the density of sites with $x_i\neq y_i$ scales for $t\to\infty$ as $(p_c-p)^\beta$ with $\beta=0.34\pm0.01$, while it decays for $p>p_c$ with a characteristic time $T$ which scales as $T\sim(p-p_c)^{-\nu_\parallel}$ with $\nu_\parallel=1$. Since these exponents disagree grossly with the DP values $\beta=0.2765$, $\nu_\parallel=1.7338$ [25], it was concluded that this transition is not in the DP universality class.

While a second order synchronization transition certainly exists in this model, details are flawed for several reasons, and we claim that the transition is in the DP class. The first problem is that it is notoriously difficult to measure $\beta$ directly in DP and similar processes, due to the very slow convergence towards the stationary state. At $p=p_c$, the density of ''active'' sites in DP (corresponding to sites with $x_i\neq y_i$ in the present model) scales as $\rho\sim t^{-\delta}$ with $\delta=0.1595$ [25]. Near $p_c$, this means that one has to wait excessively long until the stationary state is reached. Much more reliable results are obtained by following the approach towards the asymptotic state, e.g., by measuring the decay of $\rho$ with time.

The second problem is that rule 18 is well known to have very slow convergence towards its asymptotic state [26], in contrast to claims made in [17]. Therefore the strategy of

 

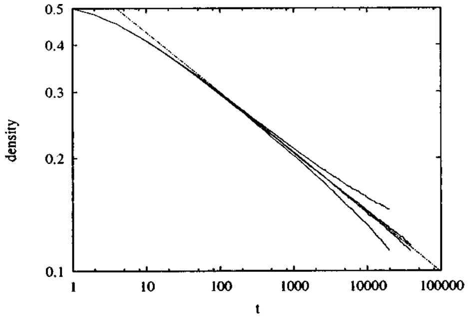


FIG. 1. Log-log plot of $\rho(t)$ for rule 90, for several values of *p*: 0.1902, 0.190 59, 0.190 65, 0.1910 from top to bottom. Statistical errors are smaller than the thickness of the lines. The straight dashed line has slope $-0.1595$, as predicted by DP.

discarding a transient of a few hundred time steps used in [17] is bound to induce errors. When starting with a random initial state, rule 18 orders into domains in which $x_i^t$ is zero either for even $i+t$ or for odd $i+t$. The boundaries between these domains move according to annihilating random walks, so that the domain sizes grow $\sim\sqrt{t}$ and the density of domain walls decreases as $1/\sqrt{t}$. Asymptotically, the entire lattice is one single domain. On the sublattice where $x_i^t$ is not identically zero, its evolution follows the ''additive'' rule 90 given by $f(0,0,1)=f(1,0,0)=f(0,1,1)=f(1,1,0)=1$. The invariant state of the latter is completely random. Therefore, rule 90 must show the same synchronization threshold as rule 18 and the same critical exponents, but it involves *no* transient whatsoever if one starts with random initial conditions.

The density $\rho(t)$ for rule 90 is shown in Fig. 1 for several values of *p*. To obtain these data we used circa 1000 lattices of size $L=10\,000$ for each *p*, which gave a sample more than 100 times larger that that of [17]. We see clearly a power behavior for large *t* (with strong small-*t* corrections) for $p=p_c=0.190\,61\pm0.000\,03$. This is quite far from the value given in [17] and implies immediately that the insets in Figs. 2 and 3 of that paper are very misleading. We also see from Fig. 1 that $\delta=0.159\pm0.003$ in excellent agreement with DP. After having determined $p_c$ in this way, we performed very long runs (with up to 400 000 time steps and with *L* up to 40 000) for $p<p_c$ in order to estimate $\rho(t=\infty)$. Such long runs were needed, since otherwise we would have suffered from systematic errors. Results are shown in Fig. 2 and give $\beta=0.277\pm0.007$, again in perfect agreement with DP [27].

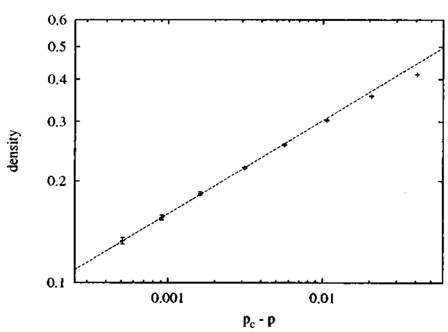


FIG. 2. Log-log plot of $\rho(t=\infty)$ against $p_c-p$, with $p_c$ as obtained from Fig. 1. The dashed line has slope 0.2765, as predicted by DP.

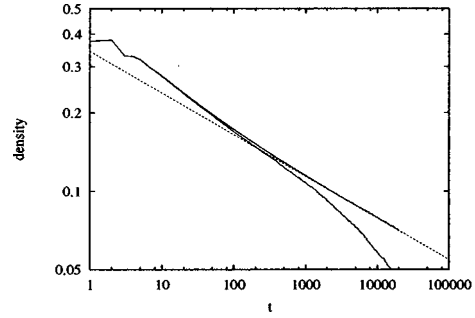


FIG. 3. Same as Fig. 1, but for rule 18 and for two values of *p* only: 0.1906 (top) and 0.192 (bottom). The first gives perfect agreement with DP for large *t* but large deviations at small *t*. The second would give a better least square fit to a straight line for $10<t<5000$, but this would yield wrong estimates of $p_c$ and $\delta$.

Finally, we show in Fig. 3 the density decay $\rho(t)$ for rule 18, with random initial conditions and without discarding any transient. We see indeed rather large corrections to scaling for times $\gg 10^2$. If we would locate the critical point by means of least square fits including the short time region, we would systematically overestimate $p_c$ and $\delta$, just as was done in [17].

We also studied rule 22. This is not an additive rule, and it has a nontrivial invariant measure with maybe zero entropy, but nonzero Lyapunov exponent [28] (for the notions of chaos, entropy and Lyapunov exponents for CA, see [29]). Even if the entropy is not zero, there are very long ranged correlations in the invariant measure of rule 22 [28]. It is thus of interest to see whether there is still a synchronization transition, and whether it is still in the DP universality class. We found again perfect agreement with DP. The critical point is at $p_c=0.227\,35\pm0.000\,05$. It is easily seen that the sum of right- and left-moving Lyaponov exponents has to be positive for $p_c$ to be non-zero in any one-dimensional CA. But the above values for $p_c$ and the known values for the Lyaponov exponents for rules 22 and 90 [28,29] suggest that there is no simple relationship beyond this qualitative criterion.

On the theoretical side we also have no formal proof of the universality with DP, but we can use exactly the same heuristic arguments which were used in [30] to argue that damage spreading transitions are generically in the DP universality class. We refer to [30] for a detailed discussion, including caveats and limitations of the expected universality. Our present results underline again the remarkable robustness of DP critical behavior. In contrast to a statement made in [17], up to now DP universality was verified in all tested cases (even if the original authors often found violations, such as in the present case), provided the criteria listed in [30] were met.

Let us now briefly discuss systems with continuous variables such as coupled map lattices. The main difference between these and CA is that synchronization is never perfect for finite time, even if $\epsilon>\epsilon_c$. Instead, the differences $|x_i^t-y_i^t|$ decrease exponentially with *t* when the systems synchronize. But this means that close to threshold statistical or chaotic fluctuations can make the system desynchronize again, at least locally. Technically spoken, the system does not enter an absorbing state when it synchronizes locally, in contrast to the discrete case. This implies very different scaling exponents, as first noticed in [21] and verified in [20,22,23]. The generic stochastic partial differential equation with these features contains a diffusion term, a local nonlinear term, and a multiplicative noise term. The logarithm of the field appearing in this stochastic PDE satisfies the KPZ equation with an additional term, which prevents the height variable from overcoming a barrier that we can conveniently place at $h=0$. The synchronization transition in this version corresponds to a transition from a surface pinned at $h\approx 0$ (desynchronized state), to a surface drifting towards $h=-\infty$ (synchronized state).

We conjecture that transitions found in neural network models similar to that of [18] but with local couplings are in this universality class. The precise model of of [18] is not in this class since it has a layered structure with long range couplings in one direction. In the complementary directions one should therefore expect mean field type correlations. It would be interesting to make detailed simulations of the model of [18] and of similar models with local couplings to verify this numerically.

On the other hand, we conjecture that the synchronization transition studied in [19] is *not* an this universality class, although it involves only local interactions. In [19] the coupling strength was called $\gamma$, and perfect synchronization occurred exactly at $\gamma=1$. As seen from Eq. (1) of [19], the equivariance group of the coupled system changes at $\gamma=1$. For $\gamma\neq 1$ the system is invariant under phase transformations $A_{1,2}\rightarrow A_{1,2}e^{i\phi_{1,2}}$ and under the exchange $A_1\leftrightarrow A_2$. For $\gamma=1$ one has the additional symmetry under phase rotations $A_1\pm A_2\rightarrow(A_1\pm A_2)e^{i\phi_\pm}$. As stressed in [31], (de)synchronization is essentially a phenomenon of spontaneous symmetry breakdown. As in other transitions with spontaneous symmetry breaking, the universality class of such transitions should depend crucially on the type of symmetry broken, and should be particularly sensitive to symmetry changes at the critical point.

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